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Bulk versus boundary (gravitational Casimir) effects in quantum creation of inflationary brane-world Universe

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ABSTRACT

The role of bulk matter quantum effects (via the corresponding effective potential discussed on the example of conformal scalar) and of boundary matter quantum effects (via the conformal anomaly induced effective action) is considered in brane-world cosmology. Scenario is used where brane tension is not free parameter, and the initial bulk-brane classical action is defined by some considerations. The effective bulk-brane equations of motion are analyzed. The quantum creation of 4d de Sitter or Anti-de Sitter(AdS) brane Universe living in 5d AdS space is possible when quantum bulk and (or) brane matter is taken into account. The consideration of only conformal field theory (CFT) living on the brane admits the full analytical treatment. Then bulk gravitational Casimir effect leads to deformation of 5d AdS space shape as well as of shape of spherical or hyperbolic branes. The generalization of above picture for the dominant bulk quantum gravity naturally represents such scenario as self-consistent warped compactification within AdS/CFT set-up.

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1 Introduction

The idea that observable Universe represents the brane where 4d gravity is trapped [1] embedded in higher dimensional bulk space attracted enormous attention. The study of cosmological aspects of such brane-worlds [2, 3] (and refs. therein) indicates towards the possibility to construct the inflationary brane Universe [2]. Some observational manifestations of bulk matter fields may occur on the brane level [4]. As brane-world Universe is naturally realized in more than four dimensional bulk space it could be related with recent studies of AdS/CFT correspondence [5]. One attempt could be in implementing of RS warped compactification within the context of RG flow in AdS/CFT set-up. It may be done in the simplest form in the way suggested in refs.[8, 6] via consideration of quantum CFT living on the brane.

Indeed, this scenario significally varies from original approach where brane tension (brane cosmological constant) is free parameter of theory permitting actually the existence of brane solution by its fine-tuning. In scenario [8, 6] the action is defined from the beginning and surface terms are added in closed manner. The role of surface terms is to make the variational procedure to be well-defined and to cancell the leading divergences of the action around AdS bulk space. Hence, brane tension is not free parameter anymore. However, its role is taken by quantum effects. Indeed, quantum effects of brane CFT induce the conformal anomaly and anomaly induced effective action. This 4d anomaly induced (brane) effective action should be added to the complete effective action of five-dimensional theory. In fact, it gives explicit contribution to brane tension permitting to have the consistent curved brane solutions. As the result, the possibility of quantum creation of de Sitter or Anti-de Sitter brane living in 5d AdS Universe has been proved in refs.[8, 6, 7]. The simplest choice for brane CFT is maximally supersymmetric (SUSY) Yang-Mills theory. Thus, conformal anomaly of brane CFT induces the brane effective tension which is responsible for de Sitter (or AdS) geometry of brane. Note that anomaly induced effective action may be considered as kind of gravitational Casimir effect [9] (for a recent introduction to Casimir effect, see [10]).

Developing further the study of warped compactifications with curved boundary (inflationary brane) within AdS/CFT correspondence the natural question is about the role of quantum bulk effects in such scenario. In other words, quantum effects of five-dimensional bulk gravity (i.e. bulk gravita-

tional Casimir effect) should be taken into account. When the boundary of AdS space is flat it can be done [11] in the analogy with the usual calculation of quantum effective action in Kaluza-Klein multi-dimensional gravity [12] (for a review and complete list of references, see [14]). However, for AdS bulk space with curved boundary such quantum gravity calculation is much more involved. This will be done elsewhere.

In the present paper, in order to estimate (at least, qualitatively) the role of bulk quantum effects to the scenario of refs.[8, 6, 7] we consider the contribution of quantum bulk matter (on the example of scalar) to complete five-dimensional effective action. Having the structure of bulk effective action for conformal matter we discuss brane-world cosmology where bulk and boundary quantum effects are taken into account. The effective bulk-brane equations of motion are derived and their solutions are analyzed. It is shown that due to such effects the quantum creation of de Sitter or Anti-de Sitter brane living in 5d AdS Universe is possible, where bulk quantum effects deform the shape of constant curvature brane (our observable Universe). Note that such Universe occurs when only brane quantum effects are included. Hence, bulk quantum effects modify the geometrical configuration which is recovered in their absence. It is interesting that above quantum creation is possible only due to bulk matter quantum effects,i.e. when no CFT lives on the brane.

2 Gravitational Casimir effect for bulk scalar in AdS space

Let us start from the following lagrangian (Euclidean sector) for a conformally invariant massless scalar field with scalar-gravitational coupling

$$\mathcal{L} = \sqrt{g}\chi(-\Box + \xi R)\chi \tag{1}$$

where $\xi = \frac{\mathcal{D}-2}{4(\mathcal{D}-1)}$. Having in mind the applications to Randall-Sundrum scenario one takes $\mathcal{D}$ to be odd. First of all one considers the warped metric of the form typical for warped compactification [1]:

$$ds^2 = dy^2 + \mathrm{e}^{2A(y)} d\Omega^2_{\mathcal{D}-1} = \mathrm{e}^{2A(z)} \left[dz^2 + d\Omega^2_{\mathcal{D}-1} \right] , \tag{2}$$

where $d\Omega^2_{\mathcal{D}-1}$ corresponds to a $\mathcal{D}$-1-dimensional constant curvature symmetric space $\mathrm{M}_{\mathcal{D}-1}$, namely $\mathrm{R}_{\mathcal{D}-1}$, $\mathrm{S}_{\mathcal{D}-1}$ and $\mathrm{H}_{\mathcal{D}-1}$, the Euclidean space, the sphere and the hyperbolic space respectively. Furthermore, the warp factor is given by

$$\mathrm{e}^{A(z)} = \frac{l}{z}\,, \tag{3}$$

l being related to the cosmological constant. We may put $l = 1$ and later by dimensional analysis recover it. Furthermore, in the following, let us consider $\mathcal{D} = 5$. It may be convenient to make the conformal transformation for the metric

$$(ds^2)' = z^2 ds^2 \tag{4}$$

and for the scalar field $\chi' = z^{-3/2}\chi$. Then, for rescaled scalar χ', one gets the Lagrangian

$$\mathcal{L} = \chi'(-\Box' + \xi R_{(4)})\chi'\,, \tag{5}$$

where $R_{(4)}$ is the constant scalar curvature of M_4 and $\Box'$ is the Laplace operator in the product space $\mathrm{R} \times \mathrm{M}_4$, whose domain is subject to suitable boundary conditions at $z = l$ and $z = L$, induced by the orbifold nature of the space-time we have started with.

Now, one should calculate the one-loop effective potential, *i.e.* the effective action divided by the whole volume

$$V = \frac{1}{2L\mathrm{Vol}(\mathrm{M}_4)} \log\det\left(L_5/\mu^2\right)\,, \tag{6}$$

where

$$L_5 = -\partial_z^2 - \Box_{(4)} + \xi R_{(4)} = L_1 + L_4\,, \tag{7}$$

on $\mathrm{R} \times \mathrm{M}_4$ limited by two branes subject to boundary conditions. At the end, if L is the branes separation, we will take the limit L goes to infinity.

Since, one is dealing with a product space, the heat-kernel for L_5 is given by

$$K_t(L_5) = K_t(L_1)K_t(L_4)\,, \tag{8}$$

where

$$K_t(L_1) = \sum_n \mathrm{e}^{-t\lambda_n^2} \tag{9}$$

is the heat-kernel related to the one-dimensional operator $L_1 = -\partial_z^2$ whose domain contains the conformally transformed orbifold boundary conditions, λ_n the associated eigenvalues and

$$K_t(L_4) = \sum_\alpha \mathrm{e}^{-t\lambda_\alpha^2} \tag{10}$$

is the heat-kernel of the Laplace-like operator L_4 on M_4, $\lambda_\alpha = \mu_\alpha + \xi R_{(4)}$, μ_α being the eigenvalues of $-\Box_{(4)}$.

If we make use of the zeta-function regularization, one needs the analytical continuation of the zeta-function

$$\zeta(s|L_5) = \sum_\alpha \sum_{n=1}^\infty (\lambda_n + \lambda_\alpha^2)^{-s} \,. \tag{11}$$

Generally speaking, we do not know explicitly the spectrum of L_1. However, it is known the short t asymptotics, which is given by

$$K_t(L_1) = \sum_{r=0}^\infty K_r(L_1) t^{\frac{r-1}{2}} = \frac{L}{2\sqrt{\pi t}} + K_1(L_1) + O(t^{1/2}) + O\left(\frac{1}{L}\right) + O(\mathrm{e}^{-\frac{L^2}{t}})\,. \tag{12}$$

Here L is the brane separation, the leading term is the Weyl term and the next term is the first non-trivial boundary term, which, for dimensional reasons, is a numerical constant. Since there is no potential term, the other boundary terms are, again for dimensional reasons, of order $O(1/L)$. Thus, if L goes to infinity, the above asymptotics becomes almost exact. As a result, we have

$$\zeta(s|L_5) = \frac{L\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma(s)} \zeta\left(s - \frac{1}{2}\middle| L_4\right) + K_1(L_1)\zeta(s|L_4) + O\left(\frac{1}{L}\right)\,, \tag{13}$$

where $\zeta(s|L_4)$ is the zeta-function associated with the Laplace-like operator on M_4.

It should be noted that

$$\zeta(0|L_5) = K_1(L_1)\zeta(0|L_4)\,. \tag{14}$$

Thus, there exists also a non trivial contribution coming from the Jacobian related to the conformal transformation we have performed. We are neglecting for the moment this contribution as we argue later on that it is negligible.

As a consequence, in the large L limit, the effective potential reads

$$\begin{aligned} V &= -\frac{1}{2L\mathrm{Vol}(\mathrm{M}_4)}[\zeta'(0|L_5) + \ln \mu^2 \zeta(0|L_5)] \\ &= \frac{\sqrt{\pi}}{2\mathrm{Vol}(\mathrm{M}_4)} \zeta\left(-\frac{1}{2}\middle| L_4\right) + O\left(\frac{1}{L}\right) , \end{aligned} \tag{15}$$

In this limit, the effective potential, reduces to the Casimir energy (vacuum energy) related to the Laplace-like operator on M_4. This Casimir energy has been calculated in several places (see for example, [15, 16, 17, 18] and references therein). We recall the corresponding results.

First, in the case of flat brane R_4, the Casimir energy goes like $(O(1/L^4)$, thus it is negligible.

For the spherical brane S_4 (with radius $\mathcal{R}$), the starting point is

$$\zeta(s|L_4) = g(s)\mathcal{R}^{2s} \tag{16}$$

where

$$g(s) = \frac{1}{6}\sum_{l=1}^{\infty}(l+1)(l+2)(2l+3)\left(l^2 + 3l + \frac{9}{4}\right)^{-s} , \tag{17}$$

the analytical continuation can be easily done, due to conformal coupling in 5 dimensions and the result is

$$\zeta(s|L_4) = \frac{\mathcal{R}^{2s}}{3}\left[\zeta_H\left(2s-3, \frac{3}{2}\right) - \frac{1}{4}\zeta_H\left(2s-1, \frac{3}{2}\right)\right] , \tag{18}$$

where $\zeta_H(s,a)$ is the Hurwitz zeta-function. Thus,

$$\zeta\left(-\frac{1}{2}\middle| L_4\right) = \frac{1}{3\mathcal{R}}\left[\zeta_H\left(-4, \frac{3}{2}\right) - \frac{1}{4}\zeta_H\left(-1, \frac{3}{2}\right)\right] . \tag{19}$$

Making use of

$$\zeta_H(-m,a) = -\frac{B_{m+1}(a)}{m+1} , \tag{20}$$

where $B_n(x)$ is a Bernoulli polynomial, one gets $\zeta(-\frac{1}{2}|L_4) = 0$. In this case $V = 0$. This is also consistent with the result reported in ref. [18]. Note that taking non-conformal coupling constant in the initial Lagrangian changes qualitatively this result, then potential will not be zero anymore. It could be also non-zero for another matter fields.

In the hyperbolic brane H_4, one has [17]

$$\frac{\zeta(s|L_4)}{\mathrm{Vol}(\mathrm{H}_4)} = -\frac{\mathcal{R}^{2s-4}}{4\pi^2}\int_0^\infty \lambda^{-2s+1}\frac{(\lambda^2+\frac{1}{4})}{\mathrm{e}^{2\pi\lambda}+1}d\lambda\,. \tag{21}$$

Thus, the effective potential reads

$$V = -\frac{1}{8\pi^{3/2}\mathcal{R}^5}\int_0^\infty \frac{(\lambda^2+\frac{1}{4})}{\mathrm{e}^{2\pi\lambda}+1}d\lambda\,. \tag{22}$$

The integral can be easily evaluated. One has

$$V = -\frac{1}{64\pi^{5/2}\mathcal{R}^5}\left[\ln 2 + \frac{3}{4\pi^2}\zeta_R(3)\right]\,. \tag{23}$$

Note that back conformal transformation of this potential should be done, to recover the potential in the original AdS space. This transformation introduces the factor e^{-5A} in the above expression.

With regard to the Jacobian factor, one may introduce the interpolating

$$(ds_q^2) = z^{2q}ds^2 \tag{24}$$

with q a real parameter such that if $q=0$, then $(ds_0^2) = ds^2$ and if $q=1$, then $(ds_1^2) = (ds^2)'$. Then (see, for example, [17])

$$\ln J(g,g') = \int_0^1 eq\int_l^L dz z^{5(q-1)}\int \sqrt{g_4}d^4x\zeta(0|L_5(q))(z,x)\,, \tag{25}$$

with

$$L_5(q) = -\Box_{(5)}(q) + \xi R_{(5)}(q) \tag{26}$$

the conformal scalar operator in the interpolating metric and $\zeta(0|L_5(q))(z,x) = \frac{K_5}{(4\pi)^{5/2}}$ is the local zeta-function. In a 5-dimensional manifold without boundary, $K_5 = 0$. In our case, however, we have orbifold boundary conditions and the coefficient K_5 is not vanishing. It has been recently computed for Robin boundary condition in [19]. Unfortunately the expression given in the above reference is difficult to use in our case.

We may assume that $K_5(q)$ is a linear combination of terms

$$\sum_l a_l q^l + b_l(1-q)^l\,. \tag{27}$$

Then, we have

$$\ln J(g,g') = \sum_l \int_0^1 dq \int_l^L dz z^{5(q-1)} (a_l q^l + b_l (1-q)^l) \,. \tag{28}$$

This integral gives contributions proportional to $li(L)$, where $li(z)$ is the logarithmic integral. For L very large, the logarithmic integral diverges as $O(\frac{L}{\ln L})$. Then, it gives a contribution of order $O(\frac{1}{\ln L})$ to the effective potential.

Thus, we presented the explicit example of evaluation of gravitational Casimir effect (effective potential) for conformal bulk scalar on five-dimensional AdS space with 4-dimensional sphere or hyperboloid as a boundary. This calculation gives us an idea about the structure of effective action due to bulk matter quantum effects.

3 Quantum creation of de Sitter (Anti-de Sitter) brane-world Universe

We consider the spacetime whose boundary is four-dimensional sphere S_4, which can be identified with a D3-brane or four-dimensional hyperboloid H_4. The bulk part is given by 5 dimensional Euclidean Anti-de Sitter space AdS_5

$$ds^2_{\mathrm{AdS}_5} = dy^2 + \sinh^2 \frac{y}{l} d\Omega^2_4 \,. \tag{29}$$

Here $d\Omega^2_4$ is given by the metric of S_4 or H_4 with unit radius. One also assumes the boundary (brane) lies at $y = y_0$ and the bulk space is given by gluing two regions given by $0 \leq y < y_0$ (see[6] for more details.)

We start with the action S which is the sum of the Einstein-Hilbert action S_{EH}, the Gibbons-Hawking surface term S_{GH} [13], the surface counter term S_1 and the trace anomaly induced action W[4]:

$$\begin{aligned} S &= S_{\mathrm{EH}} + S_{\mathrm{GH}} + 2S_1 + W & (30)\\ S_{\mathrm{EH}} &= \frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(R_{(5)} + \frac{12}{l^2}\right) & (31)\end{aligned}$$

[4]For the introduction to anomaly induced effective action in curved space-time (with torsion), see section 5.5 in [14].

$$
\begin{aligned}
S_{\rm GH} &= \frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu && (32)\\
S_1 &= -\frac{3}{8\pi G}\int d^4x\sqrt{g_{(4)}} && (33)\\
W &= b\int d^4x\sqrt{\widetilde g}\widetilde F A \\
&\quad +b'\int d^4x\sqrt{\widetilde g}\left\{A\left[2\widetilde\Box^2 + \widetilde R_{\mu\nu}\widetilde\nabla_\mu\widetilde\nabla_\nu - \frac{4}{3}\widetilde R\widetilde\Box^2 + \frac{2}{3}(\widetilde\nabla^\mu\widetilde R)\widetilde\nabla_\mu\right]A \right.\\
&\quad \left. +\left(\widetilde G - \frac{2}{3}\widetilde\Box\widetilde R\right)A\right\}\\
&\quad -\frac{1}{12}\left\{b'' + \frac{2}{3}(b+b')\right\}\int d^4x\sqrt{\widetilde g}\left[\widetilde R - 6\widetilde\Box A - 6(\widetilde\nabla_\mu A)(\widetilde\nabla^\mu A)\right]^2 && (34)
\end{aligned}
$$

Here the quantities in the 5 dimensional bulk spacetime are specified by the suffices $_{(5)}$ and those in the boundary 4 dimensional spacetime are by $_{(4)}$. The factor 2 in front of S_1 in (30) is coming from that we have two bulk regions which are connected with each other by the brane. In (32), n^μ is the unit vector normal to the boundary. In (34), one chooses the 4 dimensional boundary metric as

$$
{g_{(4)}}_{\mu\nu} = \mathrm{e}^{2A}\tilde g_{\mu\nu} \tag{35}
$$

and we specify the quantities with $\tilde g_{\mu\nu}$ by using $\tilde{\ }$. G ($\tilde G$) and F ($\tilde F$) are the Gauss-Bonnet invariant and the square of the Weyl tensor [5]

$$
\begin{aligned}
G &= R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}\\
F &= \frac{1}{3}R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl}\ , && (36)
\end{aligned}
$$

In the effective action (34), with N scalar, $N_{1/2}$ spinor, N_1 vector fields, N_2 ($=0$ or 1) gravitons and $N_{\rm HD}$ higher derivative conformal scalars, b, b' and

[5] We use the following curvature conventions:

$$
\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu}\\
R_{\mu\nu} &= R^\lambda{}_{\mu\lambda\nu}\\
R^\lambda{}_{\mu\rho\nu} &= -\Gamma^\lambda_{\mu\rho,\nu} + \Gamma^\lambda_{\mu\nu,\rho} - \Gamma^\eta_{\mu\rho}\Gamma^\lambda_{\nu\eta} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\rho\eta}\\
\Gamma^\eta_{\mu\lambda} &= \frac{1}{2}g^{\eta\nu}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)\ .
\end{aligned}
$$

b'' are

$$\begin{aligned}
b &= \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\rm HD}}{120(4\pi)^2} \\
b' &= -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\rm HD}}{360(4\pi)^2} \ , \\
b'' &= 0 \ .
\end{aligned} \tag{37}$$

As usually, b'' may be changed by the finite renormalization of local counterterm in gravitational effective action. As we shall see later, the term proportional to $\left\{b'' + \frac{2}{3}(b + b')\right\}$ in (34), and therefore b'', does not contribute to the equations of motion. For $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory $b = -b' = \frac{N^2-1}{4(4\pi)^2}$. As one can see until this point the discussion repeats the one presented in ref.[7] where more detail may be found. It is interesting to note that the contribution from brane quantum gravity may be taken into account via the correspondent coefficient in above equation.

We should also note that W in (34) is defined up to conformally invariant functional, which cannot be determined from only the conformal anomaly. The conformally flat space is a pleasant exclusion where anomaly induced effective action is defined uniquely. However, one can argue that such conformally invariant functional gives next to leading contribution as mass parameter of regularization may be adjusted to be arbitrary small (or large).

The metric of S_4 with the unit radius is given by

$$d\Omega_4^2 = d\chi^2 + \sin^2\chi d\Omega_3^2 \ . \tag{38}$$

Here $d\Omega_3^2$ is the metric of 3 dimensional unit sphere. If we change the coordinate χ to σ by

$$\sin\chi = \pm\frac{1}{\cosh\sigma} \ , \tag{39}$$

one obtains

$$d\Omega_4^2 = \frac{1}{\cosh^2\sigma}\left(d\sigma^2 + d\Omega_3^2\right) \ . \tag{40}$$

On the other hand, the metric of the 4 dimensional flat Euclidean space is

$$ds_{4\mathrm{E}}^2 = d\rho^2 + \rho^2 d\Omega_3^2 \ . \tag{41}$$

Then by changing the coordinate as

$$\rho = \mathrm{e}^\sigma \ , \tag{42}$$

one gets

$$ds_{4\mathrm{E}}^2 = \mathrm{e}^{2\sigma}\left(d\sigma^2 + d\Omega_3^2\right) \ . \tag{43}$$

For the 4 dimensional hyperboloid with the unit radius, the metric is

$$ds_{\mathrm{H}4}^2 = d\chi^2 + \sinh^2\chi d\Omega_3^2 \ . \tag{44}$$

Changing the coordinate χ to σ

$$\sinh\chi = \frac{1}{\sinh\sigma} \ , \tag{45}$$

one finds

$$ds_{\mathrm{H}4}^2 = \frac{1}{\sinh^2\sigma}\left(d\sigma^2 + d\Omega_3^2\right) \ . \tag{46}$$

Motivated by (29), (40), (43) and (46), one assumes the metric of 5 dimensional space time as follows:

$$ds^2 = dy^2 + \mathrm{e}^{2A(y,\sigma)}\tilde g_{\mu\nu}dx^\mu dx^\nu \ , \quad \tilde g_{\mu\nu}dx^\mu dx^\nu \equiv l^2\left(d\sigma^2 + d\Omega_3^2\right) \tag{47}$$

and we identify A and $\tilde g$ in (47) with those in (35). Then one finds $\tilde F = \tilde G = 0$, $\tilde R = \frac{6}{l^2}$ etc. Due to Eq. (47), the actions in (31), (32), (33), and (34) have the following forms:

$$\begin{aligned}
S_{\mathrm{EH}} &= \frac{l^4 V_3}{16\pi G}\int dy d\sigma \left\{\left(-8\partial_y^2 A - 20(\partial_y A)^2\right)\mathrm{e}^{4A}\right. \\
&\quad \left. + \left(-6\partial_\sigma^2 A - 6(\partial_\sigma A)^2 + 6\right)\mathrm{e}^{2A} + \frac{12}{l^2}\mathrm{e}^{4A}\right\} &(48)\\
S_{\mathrm{GH}} &= \frac{3l^4 V_3}{8\pi G}\int d\sigma \mathrm{e}^{4A}\partial_y A &(49)\\
S_1 &= -\frac{3l^3 V_3}{8\pi G}\int d\sigma \mathrm{e}^{4A} &(50)\\
W &= V_3\int d\sigma\left[b' A\left(2\partial_\sigma^4 A - 8\partial_\sigma^2 A\right)\right. \\
&\quad \left. -2(b+b')\left(1 - \partial_\sigma^2 A - (\partial_\sigma A)^2\right)^2\right] \ . &(51)
\end{aligned}$$

Here $V_3 = \int d\Omega_3$ is the volume or area of the unit 3 sphere.

As it follows from the discussion in the previous section there is also gravitational Casimir contribution due to bulk quantum fields. As one sees on the example of bulk scalar it has typically the following form $S_{\rm Csmr}$

$$S_{\rm Csmr} = \frac{cV_3}{\mathcal{R}^5}\int dy d\sigma {\rm e}^{-A} \tag{52}$$

Note that role of (effective) radius of 4d constant curvature space (after back conformal transformation as in previous section) is played by $\mathcal{R}{\rm e}^A$. Here c is some coefficient whose value and sign depend on the type of bulk field (scalar, spinor, vector, graviton, ...) and on parameters of bulk theory (mass, scalar-gravitational coupling constant, etc). In the previous section we found this coefficient for conformal scalar. In the following discussion it is more convenient to consider this coefficient to be some parameter of the theory. Then, the results are quite common and may be applied to arbitrary quantum bulk theory. We also suppose that there are no background bulk fields in the theory (except of bulk gravitational field).

Adding quantum bulk contribution to the action S in (30) one can regard

$$S_{\rm total} = S + S_{\rm Csmr} \tag{53}$$

as the total action. In (52), $\mathcal{R}$ is the radius of S_4 or H_4.

In the bulk, one obtains the following equation of motion from $S_{\rm EH}+S_{\rm Csmr}$ by the variation over A:

$$\begin{aligned} 0 \;=\; & \left(-24\partial_y^2 A - 48(\partial_y A)^2 + \frac{48}{l^2}\right){\rm e}^{4A} \\ & +\frac{1}{l^2}\left(-12\partial_\sigma^2 A - 12(\partial_\sigma A)^2 + 12\right){\rm e}^{2A} + \frac{16\pi G c}{\mathcal{R}^5}{\rm e}^{-A}\ . \end{aligned} \tag{54}$$

First, one can consider a special solution of the bulk equation (54). If one assumes that A does not depend on σ, Eq.(54) has the following form:

$$0 = \left(-24\partial_y^2 A - 48(\partial_y A)^2 + \frac{48}{l^2}\right){\rm e}^{4A} + \frac{16\pi G c}{\mathcal{R}^5}{\rm e}^{-A}\ . \tag{55}$$

Eq.(55) has the following integral:

$$E = -\frac{1}{4}\left(\frac{d\left({\rm e}^{2\tilde A}\right)}{dy}\right)^2 + \frac{1}{l^2}{\rm e}^{4\tilde A} + \frac{1}{2l^2}{\rm e}^{2\tilde A} - \frac{4\pi G c}{3\mathcal{R}^5}{\rm e}^{-\tilde A}\ . \tag{56}$$

Then if we assume $\frac{d\left(e^{2\tilde{A}}\right)}{dy} > 0$, we find the following solution in the bulk

$$y = \frac{1}{2}\int dQ \left(-E + \frac{Q^2}{l^2} + \frac{Q}{2l^2} - \frac{4\pi Gc}{3\mathcal{R}^5}Q^{-\frac{1}{2}}\right)^{-\frac{1}{2}} . \tag{57}$$

This represents an example of self-consistent warped compactification. The analysis of such solution shows that for vanishing bulk cosmological constant it goes away from AdS space. This indicates that bulk Casimir effect acts against of warped compactification.

Let us discuss the solution in the situation when scale factor depends on both coordinates:y,σ. One can find the solution of (54) as an expansion with respect to $e^{-\frac{y}{l}}$ by assuming that $\frac{y}{l}$ is large:

$$e^A = \frac{\sinh\frac{y}{l}}{\cosh\sigma} - \frac{32\pi Gcl^3}{15\mathcal{R}^5}\cosh^4\sigma e^{-\frac{4y}{l}} + \mathcal{O}\left(e^{-\frac{5y}{l}}\right) \tag{58}$$

for the perturbation from the solution where the brane is S_4 and

$$e^A = \frac{\cosh\frac{y}{l}}{\sinh\sigma} - \frac{32\pi Gcl^3}{15\mathcal{R}^5}\sinh^4\sigma e^{-\frac{4y}{l}} + \mathcal{O}\left(e^{-\frac{5y}{l}}\right) \tag{59}$$

for the perturbation from H_4 brane solution.

On the brane at the boundary, one gets the following equation:

$$\begin{aligned} 0 \;=\; & \frac{48l^4}{16\pi G}\left(\partial_y A - \frac{1}{l}\right)e^{4A} + b'\left(4\partial_\sigma^4 A - 16\partial_\sigma^2 A\right) \\ & -4(b+b')\left(\partial_\sigma^4 A + 2\partial_\sigma^2 A - 6(\partial_\sigma A)^2\partial_\sigma^2 A\right) . \end{aligned} \tag{60}$$

We should note that the contributions from S_{EH} and S_{GH} are twice from the naive values since we have two bulk regions which are connected with each other by the brane. Substituting the solutions (58) and (59) into (60), we find

$$0 \sim \frac{1}{\pi G}\left(\frac{1}{\mathcal{R}}\sqrt{1 + \frac{\mathcal{R}^2}{l^2}} + \frac{64\pi Gl^7c}{3\mathcal{R}^{10}}\cosh^5\sigma - \frac{1}{l}\right)\mathcal{R}^4 + 8b' . \tag{61}$$

for S_4 brane and

$$0 \sim \frac{1}{\pi G}\left(\frac{1}{\mathcal{R}}\sqrt{-1 + \frac{\mathcal{R}^2}{l^2}} + \frac{64\pi Gl^7c}{3\mathcal{R}^{10}}\sinh^5\sigma - \frac{1}{l}\right)\mathcal{R}^4 + 8b' . \tag{62}$$

for H_4 brane. Here the radius $\mathcal{R}$ of S_4 or H_4 is related with $A(y_0)$, if we assume the brane lies at $y = y_0$, by

$$\tilde{R} = l e^{\tilde{A}(y_0)} . \tag{63}$$

In Eqs.(61) and (62), only the leading terms with respect to $1/\mathcal{R}$ are kept in the ones coming from S_{Csmr} (the terms including c). When $c = 0$, the previous result in [6, 7] is reproduced. Eqs.(61) and (62) tell that the Casimir force deforms the shape of S_4 or H_4 since $\mathcal{R}$ becomes σ dependent. The effect becomes larger for large σ. In case of S_4 brane, the effect becomes large if the distance from the equator becomes large since σ is related to the angle coordinate χ by (39). Especially at north and south poles ($\chi = 0, \pi$), $\cosh\sigma$ diverges then $\mathcal{R}$ should vanish as in Fig.1. Of course, the perturbation would be invalid when $\cosh\sigma$ is large. Thus, we demonstrated that bulk quantum effects do not destroy the quantum creation of de Sitter (inflationary) or Anti-de Sitter brane-world Universe. Of course, analytical continuation of 4d sphere to Lorentzian signature is supposed which leads to ever expanding inflationary brane-world Universe. However, as we see the bulk quantum effects change the effective radius of 4d sphere (or 4d hyperboloid).

When $c = 0$, the solution can exist when $b' < 0$ for S_4 brane (in this case it is qualitatively similar to quite well-known anomaly driven inflation of refs.[20]) and $b' > 0$ for H_4. For S_4 brane, if $b' < 0$, the effect of Casimir force makes the radius smaller (larger) if $c > 0$ ($c < 0$). For H_4 brane, from Eq.(62) for small $\mathcal{R}$ it behaves as

$$0 \sim \frac{64 l^7 c}{3\mathcal{R}^{10}} \sinh^5\sigma + 8b' . \tag{64}$$

Then one would have solution even if $b' < 0$. (We should note σ is restricted to be positive). Of course, one cannot do any quantitative conclusion since it is assumed $\mathcal{R}$ is large when deriving (62).

We now compare the above obtained results with the case of no quantum correction W (no matter) on the brane, i.e. when bulk quantum effects are leading. Putting $b' = 0$ in (61) and (62), one gets

$$\mathcal{R}^8 \sim -\frac{128\pi G l^6 c}{8} \cosh^5\sigma \tag{65}$$

for S_4 brane and

$$\mathcal{R}^8 \sim \frac{128\pi G l^6 c}{8} \sinh^5\sigma \tag{66}$$

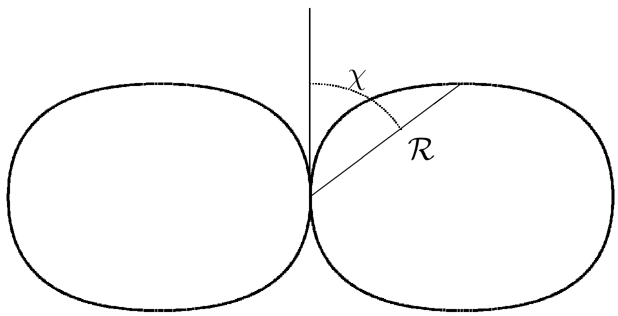

Figure 1: The rough sketch of the shape of cross section of the S_4brane when $c > 0$.

for H_4 brane. Here we only consider the leading term with respect to c, which corresponds to large $\mathcal{R}$ approximation. In case of S_4, there is no real solution for negative b' but there appears a solution for negative b' in case of H_4, where there is no solution without Casimir term $S_{\rm Csmr}$ in (52). Thus, we demonstrated that bulk quantum effects do not violate (in some cases ,even support) the quantum creation of de Sitter or Anti-de Sitter brane living in d5 AdS world.

4 Discussion

In summary, we compared the role of bulk matter and brane matter quantum effects in the realization of brane-world Universe with constant curvature 4d brane (de Sitter or Anti-de Sitter). In such Universe the bulk represents five-dimensional AdS space while observable four-dimensional Universe is the boundary (brane) of such five-dimensional space. The brane matter quantum effects may be included in the universal form, via the corresponding anomaly induced effective action on the brane. (Actually, they modify the effective brane tension which is fixed on classical level). In this way, the contribution of any specific conformally invariant matter theory (or even of brane quantum gravity) only changes the coefficients of the correspondent anomaly induced effective action. The bulk conformal matter quantum effects are more difficult to calculate. We made the correspondent evaluation for scalar in order to understand the qualitative structure of bulk effective action.

It is shown that quantum creation of inflationary (or hyperbolic) brane-

world Universe is possible. Such Universe may be induced by only bulk quantum effects , or by only brane quantum effects. When both contributions are included the role of bulk quantum effects is in the deformation of shape of 5d AdS Universe as well as of shape of 4d spherical or hyperbolic brane. (These are induced by brane CFT quantum effects).

There are few possible extensions of the presented results. As we already mentioned in the introduction, it would be really interesting (and more consistent from the AdS/CFT correspondence point of view) to investigate the role of bulk quantum gravity in such self-consistent warped Randall-Sundrum compactification with curved boundary. This is under study currently.

From another side, one can combine the current scenario for quantum induced inflationary brane-world Universe with the ones [2] where classical background matter is presented in the bulk and (or) on the brane. In any case, the number of possibilities to realize brane-world inflation occurs.

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